Duals of Inner Product Spaces

(Section 8.2)

Theorem: (Riesz Representation)

Let V be a finite-dimensional vector space over $\mathbb{R}$ (or $\mathbb{C}$). Let $\varphi: V \to \mathbb{R}$ (or $\mathbb{C}$) be a linear functional. Then $\exists\, y \in V$ such that $\forall\, x \in V$,

$$\boxed{\varphi(x) = \langle x, y \rangle}$$

Proof: Need to construct y.

Let $\{e_1, \ldots, e_n\}$ be a basis for V. Since φ is linear, if $x \in V$, $x = \sum_{i=1}^{n} \alpha_i e_i$

for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ (or $\mathbb{C}$),

then

$$\varphi(x) = \varphi\left(\sum_{i=1}^{n} \alpha_i e_i\right)$$

$$= \sum_{i=1}^{n} \varphi(\alpha_i e_i) = \sum_{i=1}^{n} \alpha_i \varphi(e_i)$$

Claim: $y = \sum_{i=1}^{n} \overline{\varphi(e_i)} e_i$

is the desired element.

$$\langle x, y \rangle = \left\langle \sum_{j=1}^{n} \alpha_j e_j , \sum_{i=1}^{n} \overline{\varphi(e_i)} e_i \right\rangle$$

$$= \sum_{j=1}^{n} \left\langle \alpha_j e_j , \sum_{i=1}^{n} \overline{\varphi(e_i)} e_i \right\rangle$$

(linearity in 1^{st} coordinate)

$$\sum_{j=1}^{n} \left\langle \alpha_j e_j, \sum_{i=1}^{n} \overline{\varphi(e_i)} e_i \right\rangle$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \left\langle \alpha_j e_j, \overline{\varphi(e_i)} e_i \right\rangle$$

(additivity in 2nd coordinate)

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_j \varphi(e_i) \langle e_j, e_i \rangle$$

(linearity in 1st coordinate, conjugate linearity in 2nd)

If, in addition, we assume $\{e_1, \dots, e_n\}$ is an orthonormal basis, then

$$\langle e_j, e_i \rangle = \delta_{i,j}$$

$$= \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

We get

$$\langle x, y \rangle = \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_j \varphi(e_i) \langle e_j, e_i \rangle$$

$$= \sum_{i=1}^{n} \alpha_i \varphi(e_i) = \varphi(x). \quad \square$$

Observations:

1) Note that for a fixed y,

$$\varphi_y(x) = \langle x, y \rangle \text{ defines}$$

a linear functional on V.

Riesz Representation says that all linear functionals arise in this manner.

2) The y obtained in Riesz Representation is unique, since if $\exists\ z \in V$,

$\varphi(x) = \langle x, z \rangle\ \forall\ x \in V$,

Then

$$\langle x, y-z \rangle = \langle x, y \rangle - \langle x, z \rangle$$
$$= \varphi(x) - \varphi(x)$$
$$= 0$$

$\Rightarrow y - z = 0_V$, so $y = z$.

3) Riesz Representation still holds in infinite dimensions, but you need a better notion of a basis and completeness of V with respect to the norm $\|x\| = \langle x, x \rangle^{1/2}$ (Math 451)

In particular, Riesz Representation will hold for $l_2(\mathbb{N})$, but you need φ to be continuous.

Example 1: Recall the linear functional

$$\varphi : \mathbb{R}^n \to \mathbb{R},$$

$$\varphi\left(\sum_{i=1}^{n} \alpha_i e_i\right) = \sum_{i=1}^{n} \alpha_i$$

for $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.

By Riesz Representation,

$\exists\, y \in \mathbb{R}^n$, $\varphi(x) = \langle x, y \rangle$

By the proof,

$$y = \sum_{i=1}^{n} \overline{\varphi(e_i)} e_i$$

$$= \sum_{i=1}^{n} \varphi(e_i) e_i \quad (\varphi(e_i) \in \mathbb{R})$$

But $\varphi(e_i) = 1 \ \forall\, 1 \le i \le n$,

so $y = \sum_{i=1}^{n} e_i = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

Adjoints

(Section 8.3)

Only for Inner Product Spaces!

Assume V is a finite-dimensional inner product space over $\mathbb{R}$ (or $\mathbb{C}$).

Let T be a linear transformation from V to V. Then $\exists$ a linear transformation $T^*: V \to V$ such that $\boxed{\langle Tx, y \rangle = \langle x, T^* y \rangle}$

$\forall x, y \in V$

T^* is called the adjoint of T.

Note by Riesz Representation, that $\langle Tx, y\rangle$ for a fixed x is a linear functional in y if V is over $\mathbb{R}$ (linearity in the second coordinate).

Define $\varphi(x) = \langle Tx, y\rangle \in V^*$.

Then by Riesz Representation,

$\exists\ z \in \mathbb{R}^n$,

$$\langle Tx, y \rangle = \varphi(x) = \langle x, z \rangle$$

Set $z = T^* y$, prove that this is linear!

Theorem: (adjoint identification)

Let $T:\mathbb{R}^n \to \mathbb{R}^n$ (or $\mathbb{C}^n \to \mathbb{C}^n$) be a linear transformation.

Then if A is the matrix of T with respect to the standard basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$), then $\forall\ y \in \mathbb{R}^n$,

$$T^* y = A^t y$$

If $y \in \mathbb{C}^n$,

$$T^* y = A^* y$$

where $(A^*)_{i,j} = \overline{A_{j,i}}$,

so A^* is the conjugate transpose of A.

Proof: Let $x, y \in \mathbb{R}^n$.

Then

$$\langle Tx, y\rangle = \langle Ax, y\rangle$$

$$= y^t A x$$ (x, y thought of as column vectors)

$$= y^t (A^t)^t x$$

$$= (A^t y)^t x$$

$$= \langle x, A^t y\rangle$$

The proof for $\mathbb{C}^n$ follows by replacing "t" with "*" at every point in the proof.

Note same rules apply:

$$(AB)^* = B^* A^*$$

$$(A^*)^* = A$$

$\square$

Theorem: (kernel and range) Let $A \in M_n(\mathbb{R})$. Then

$$\ker(A)^{\perp} = \operatorname{Ran}(A^t)$$

proof: Show containment both ways!

$\Rightarrow$ $\ker(A)^{\perp} \subseteq \operatorname{Ran}(A^t)$

By taking orthogonal complements, this is equivalent to showing

$$(\operatorname{Ran}(A^t))^{\perp} \subseteq (\ker(A)^{\perp})^{\perp} = \ker(A)$$

Let $x \in (\mathrm{Ran}(A^t))^\perp$. Then

$$\begin{aligned} \|Ax\|^2 &= \langle Ax, Ax \rangle \\ &= (Ax)^t Ax \\ &= (Ax)^t (A^t)^t x \\ &= (A^t Ax)^t x \\ &= \langle x, A^t Ax \rangle \\ &= 0 \quad \text{since } A^t Ax \in \mathrm{Ran}(A^t) \end{aligned}$$

Hence $Ax = 0$ and so $x \in \ker(A)$.

$\Leftarrow$ $\mathrm{Ran}(A^t) \subseteq \ker(A)^\perp$

Let $x \in \mathrm{Ran}(A^t)$. Then $x = A^t y$ for some $y \in \mathbb{R}^n$. Now if $z \in \ker(A)$,

$$\begin{aligned}
\langle x, z \rangle &= \langle A^t y, z \rangle \\
&= z^t A^t y \\
&= (Az)^t y \\
&= \langle y, Az \rangle \\
&= \langle y, 0 \rangle \quad \text{since } z \in \ker(A) \\
&= 0
\end{aligned}$$

and so $x \in \ker(A)^\perp$ $\square$